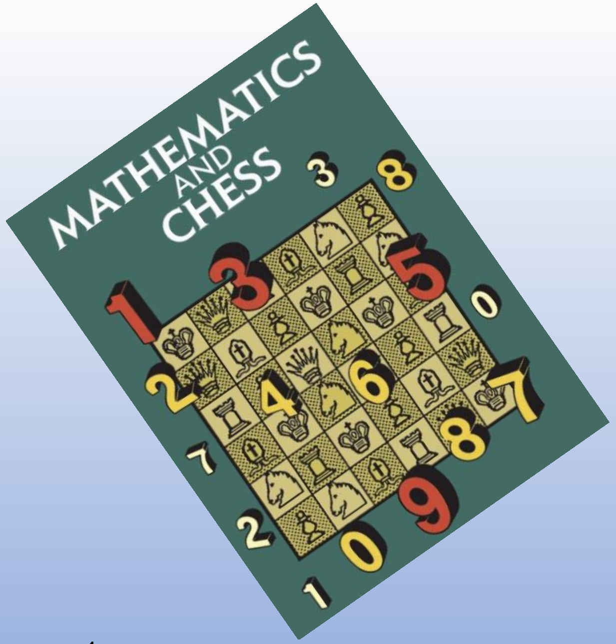
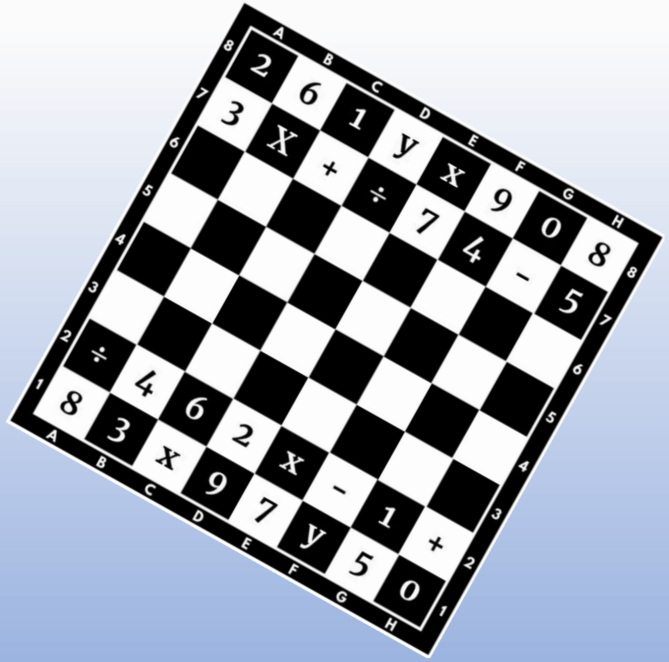
"Chess-Math" Lecture

Rilton Cup, Stockholm

December 2025

Dr. Yevgeny Levanzov





Introduction

- The lecture will be about mathematical problems related to the chessboard and chess pieces, when we will present problems of different types.
- I recommend that each of you will have a chessboard open - where you can place pieces as much as you want.
  For example: https://lichess.org/editor

What is mathematics – in a nutshell

- Although we all begin learning mathematics with arithmetic, it is important to emphasize that mathematics is not about numbers, and there are many problems and branches in it that do not deal with numbers at all (for example: certain problems in geometry).
- Mathematics deals with logically drawing conclusions about some structure from certain assumptions we have about it, as well as basic assumptions that we consider to be true (axioms).
- In our case, the structure will be the chessboard and pieces, and we would like to prove certain properties regarding them.

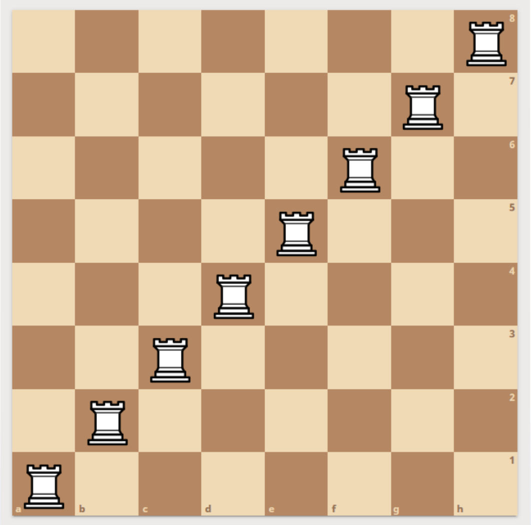
Problem #1

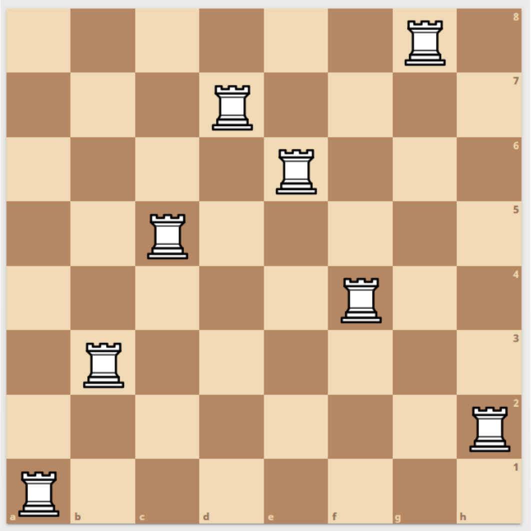
Maximum number of pieces without threatening each other

- Rooks
- Queens
- Bishops
- Kings
- Knights

- We will not solve for pawns - too weird...

Rooks - 8



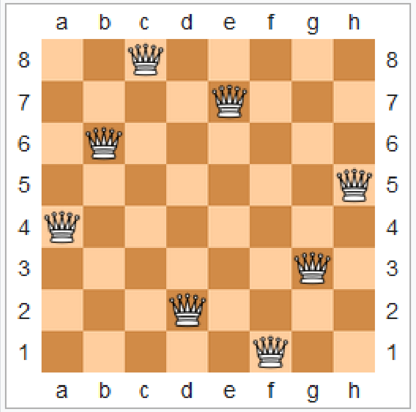


- A logical reasoning question: How can we easily show that we cannot place 9 rooks?
  - **Pigeonhole principle**
  - **An upper bound for the solution of the Queens problem**

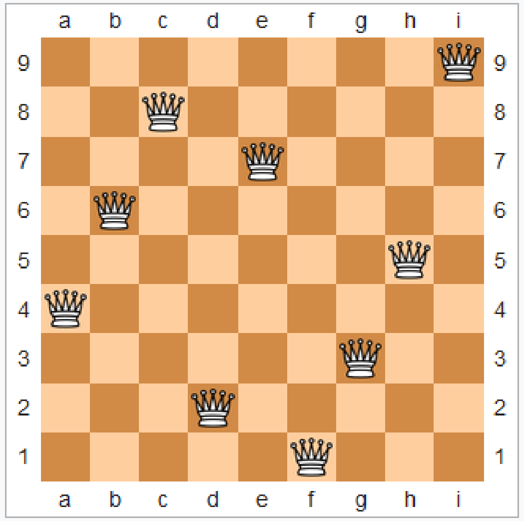
The Pigeonhole Principle

- A basic principle in combinatorics, which says the following simple thing: If we have 11 pigeons and 10 cells, then if we put the pigeons into the cells, there will be a cell with at least 2 pigeons.
- In general: if we put *n+1* pigeons into *n* cells, there will be a cell with at least 2 pigeons.
- Generalization of the principle: if we have, for example, 33 pigeons and 10 cells, then there will be a cell with at least 33/10 pigeons, that is, at least 4 (because we round up to a whole number).
- Despite of its simplicity, this principle appears in many complex problems.
- The principle was introduced by the French-German mathematician of the 19^{th} century, Dirichlet.
- Back to the rooks problem: in our case, the pigeons are rooks and the cells are ranks. Therefore, a placement of 9 rooks will result in a rank with 2 rooks.

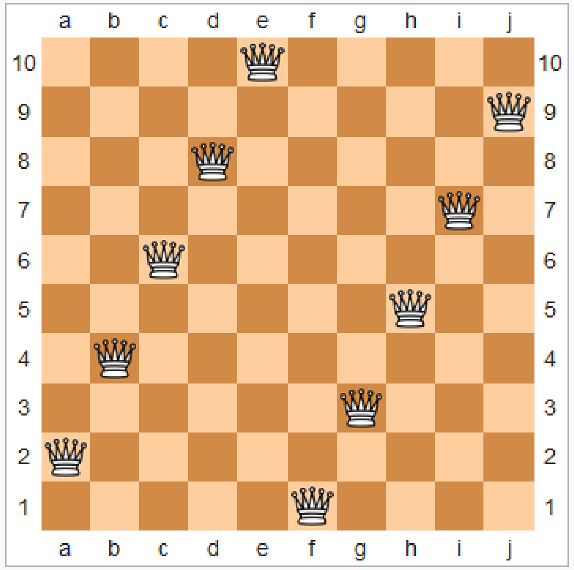
Queens - 8



Staircase solution for 8 queens



Staircase solution for 9 queens



Staircase solution for 10 queens

- A somewhat surprising fact – the answer for rooks and queens is 8, despite the difference in the strength of the pieces.
- The main motif here is the “knight-move” distance between the queens.
- It is known that there is a solution for every board starting from 4x4.
- There is an efficient algorithm for finding a solution.

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Fundamental solutions	1	0	0	1	2	1	6	12	46	92	341	1,787	9,233	45,752
All solutions	1	0	0	2	10	4	40	92	352	724	2,680	14,200	73,712	365,596

Queens - anecdotes

- Tzur Luria, Doctor of Mathematics (then in ETH Zurich), and a chess player (rated around 2100) - wrote a few years ago a paper dealing with bounding the number of solutions to the Queens problem on general nxn boards, and gave an almost tight bound.
- In 2022, Dr. Michael Simkin (now in MIT), improved Luria's work, and gave an exact bound to the number of solutions, which is approximately $(0.143n)^n$ for an nxn board.

NEW BOUNDS ON THE NUMBER OF n-QUEENS CONFIGURATIONS

ZUR LURIA

ABSTRACT. In how many ways can n queens be placed on an $n \times n$ chessboard so that no two queens attack each other? This is the famous n-queens problem. Let $Q(n)$ denote the number of such configurations, and let $T(n)$ be the number of configurations on a toroidal chessboard. We show that for every n of the form 4^k+1, $T(n)$ and $Q(n)$ are both at least $n^{\Omega(n)}$. This result confirms a conjecture of Rivin, Vardi and Zimmerman for these values of n [11]. We also present new upper bounds on $T(n)$ and $Q(n)$ using the entropy method, and conjecture that in the case of $T(n)$ the bound is asymptotically tight. Along the way, we prove an upper bound on the number of perfect matchings in regular hypergraphs, which may be of independent interest.

THE NUMBER OF n-QUEENS CONFIGURATIONS

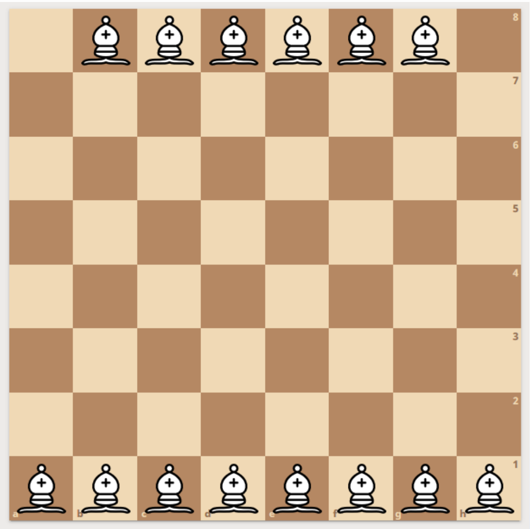
MICHAEL SIMKIN

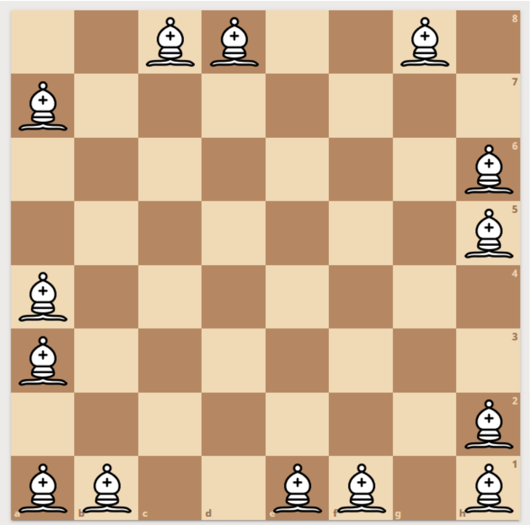
ABSTRACT. The n-queens problem is to determine $\mathcal{Q}(n)$, the number of ways to place n mutually non-threatening queens on an $n \times n$ board. We show that there exists a constant $\alpha = 1.942 \pm 3 \times 10^{-3}$ such that $\mathcal{Q}(n) = ((1 \pm o(1))ne^{-\alpha})^n$. The constant α is characterized as the solution to a convex optimization problem in $\mathcal{P}([-1/2, 1/2]^2)$, the space of Borel probability measures on the square.

The chief innovation is the introduction of limit objects for n-queens configurations, which we call *queenons*. These form a convex set in $\mathcal{P}([-1/2, 1/2]^2)$. We define an entropy function that counts the number of n-queens configurations that approximate a given queenon. The upper bound uses the entropy method of Radhakrishnan and Linial–Luria. For the lower bound we describe a randomized algorithm that constructs a configuration near a prespecified queenon and whose entropy matches that found in the upper bound. The enumeration of n-queens configurations is then obtained by maximizing the (concave) entropy function in the space of queenons.

Along the way we prove a large deviations principle for n-queens configurations that can be used to study their typical structure.

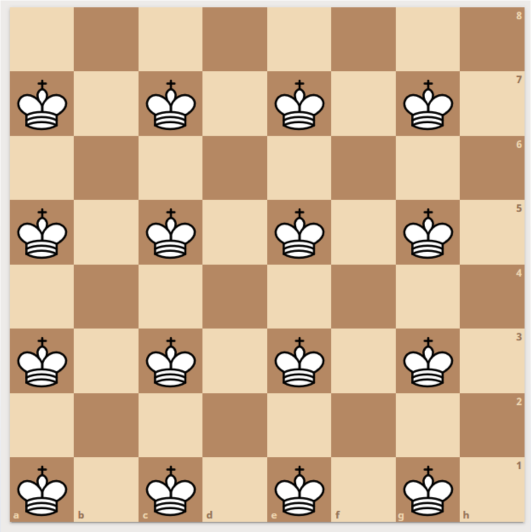
Bishops - 14

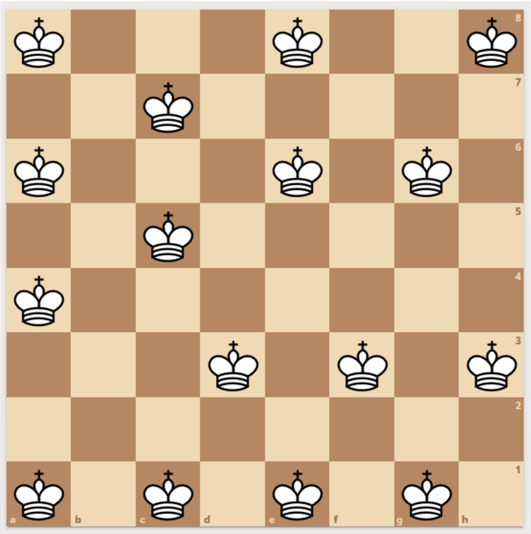




- Obviously, the solution must be an even number - because in fact we solve “separately" for each color of squares (which is not necessarily true for boards with an odd number of squares - for example an 1x1 board...)

Kings - 16



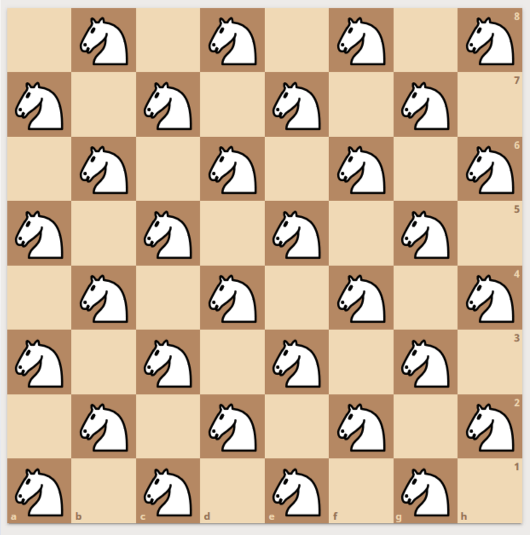


Knights - 32

- Of course there is another solution where the knights are on light squares.
- How can we show that it is not possible to mix the two colors and find additional solutions with more than 32 knights?!

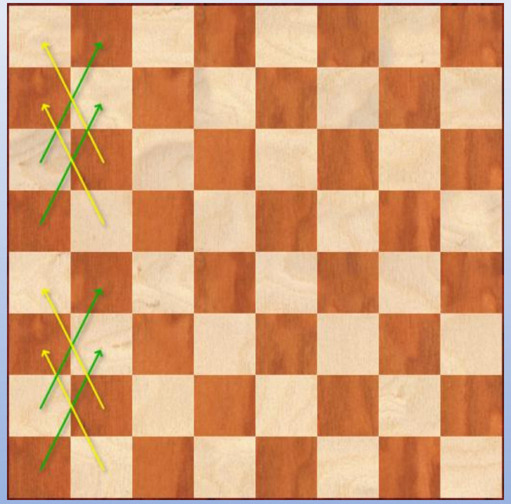
Intuition:

Let’s assume we put most of the knights on dark squares. Then the first knight we put on a light square will take away at least two dark squares (if we put it on h1, for example - it will take away f2, g3), and so it turns out that we added one more knight, but removed two – and so it seems that this is a bad strategy.



Knights – a proof that 32 is indeed the maximum

- Partition the 64 squares into 32 disjoin pairs, as follows: a1 will be paired with b3, b1 with a3, a2 with b4, b2 with a4, and so on for the rest of the board.
- Note that on the squares of each pair we can put at most one knight (from our choice – each pair of squares is knight's move away).
- Hence, we can place at most 32 knights.
- It can be shown that the only arrangements of 32 knights are the ones where all knights are on the same color.



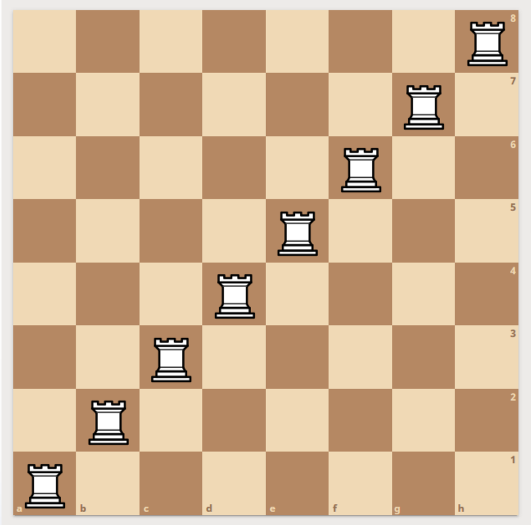
Problem #2

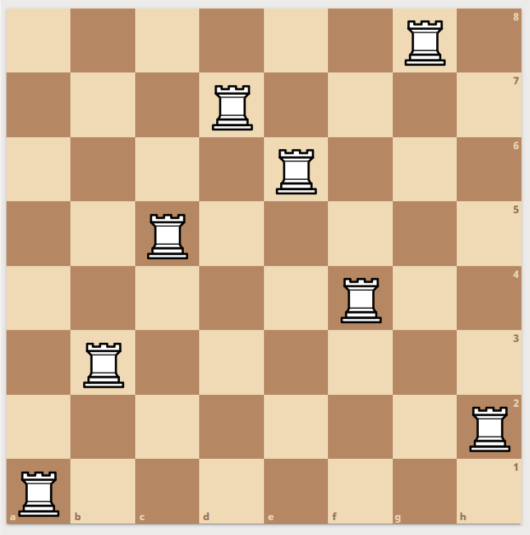
Minimum number of pieces attacking the entire board

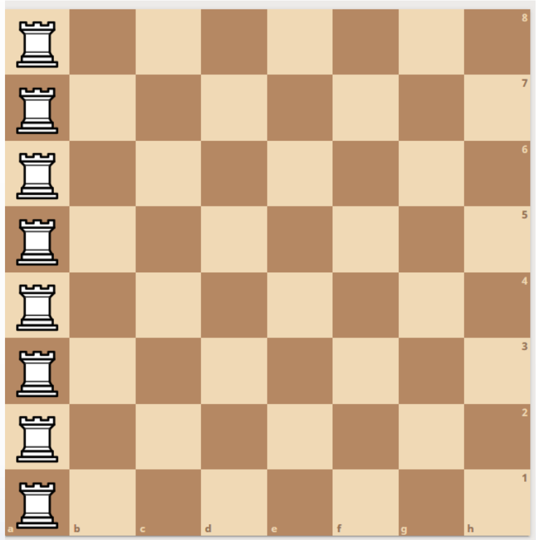
- Rooks
- Queens
- Bishops
- Kings
- Knights

- We will not solve for pawns - too weird...

Rooks - 8







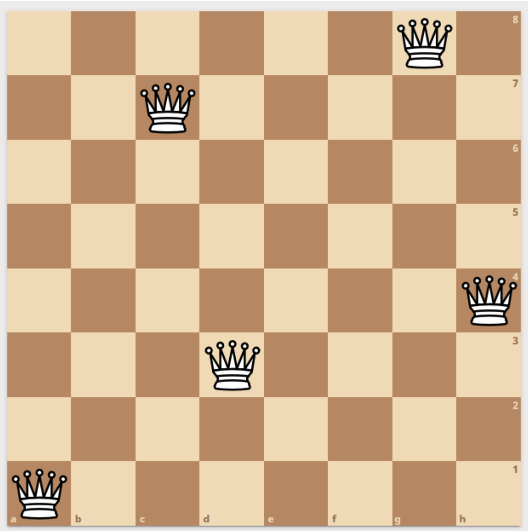
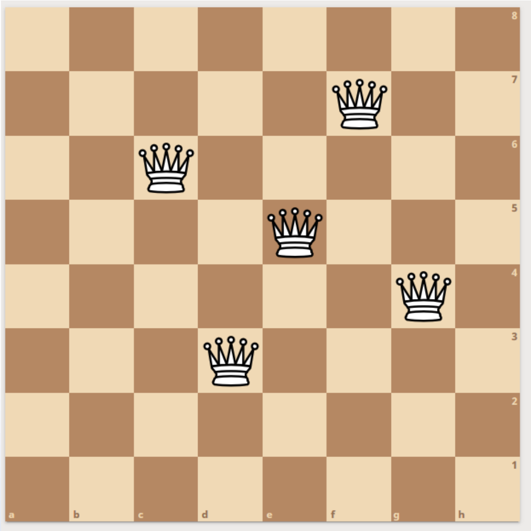
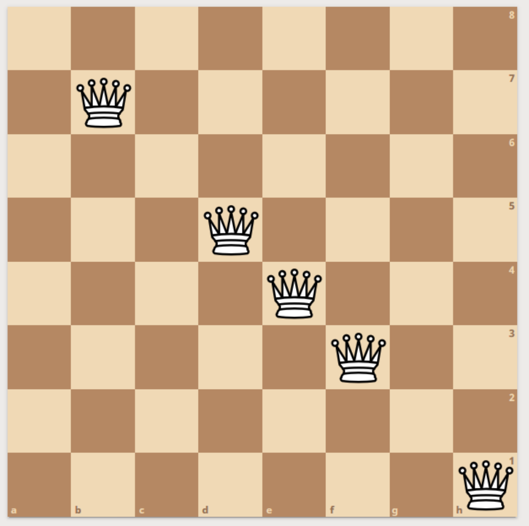
Can we formulate a clear argument - why can't we cover the entire board with only 7 rooks?

7 rooks are placed in at most in 7 different ranks - so there will always be a rank without a rook, let's say it's 6^{th} rank.

7 rooks are placed in at most in 7 different files – so there will always be a file without a rook, let's say it's e-file.

So – no rook attacks e6.

Queens - 5

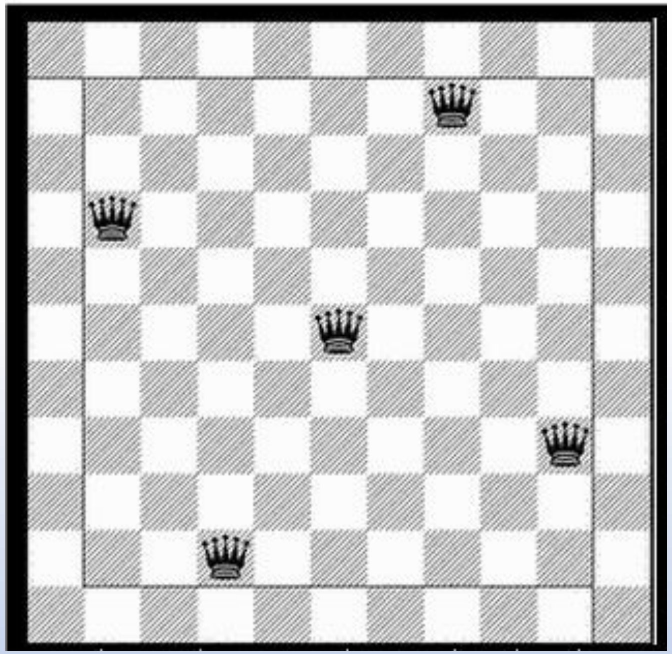




Food for thought - how many queens do we expect to need in order to control all the squares in larger boards - for example 9x9, 10x10, 11x11?

Queens - an amazing anecdote

It's hard to believe - but 5 queens are enough to attack all 121 squares on an 11x11 board (almost twice as much as a regular board!).

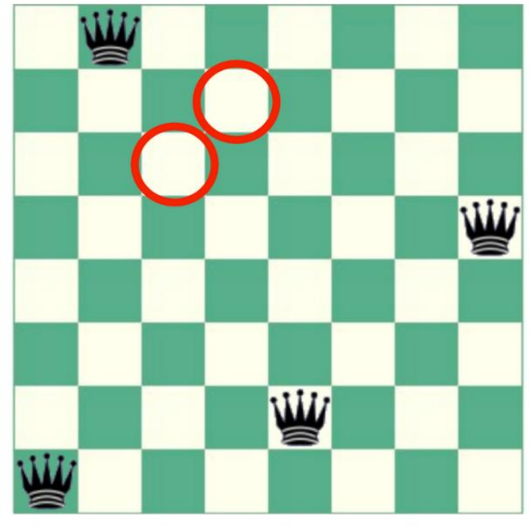
As you can see - all queens are on black squares in this solution - and it is even more surprising how they cover 60 light squares - with no light diagonals at all...



How far are 4 queens from being a solution?

- 4 queens can cover 62 squares as you can see in the board on the left - except for the two red squares - everything is covered.

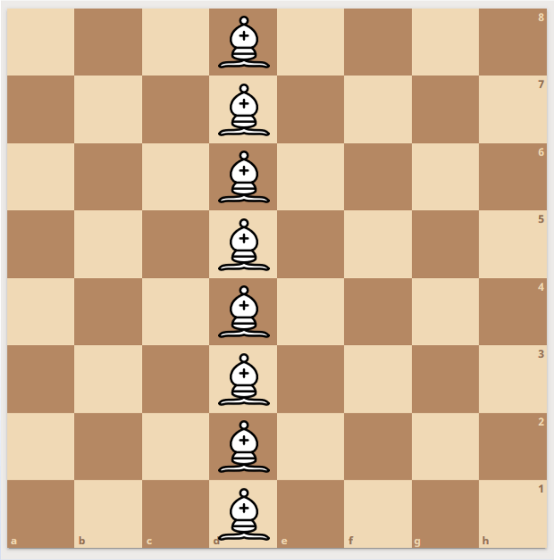
- It's interesting that at least according to a quick search I did on the internet - it seems that there is no clear logical proof of why it is impossible to cover 64 or 63 squares, but simply a code was run, checking all possible placements of 4 queens - and it turned out that 62 squares is the maximum.



Bishops - 8

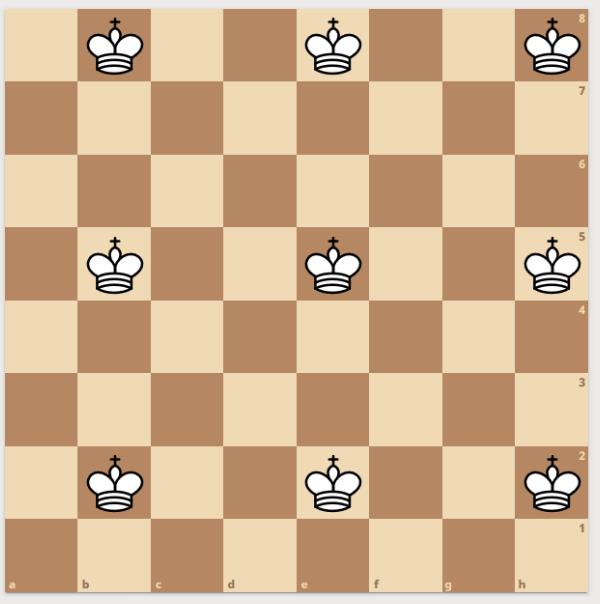
- Of course, symmetrical solutions can be formed by placing all bishops on the e-file, or on 4th, 5th ranks.

- We note that if we would put the bishops on the c-file, for example - we will no longer cover h4 and h5.



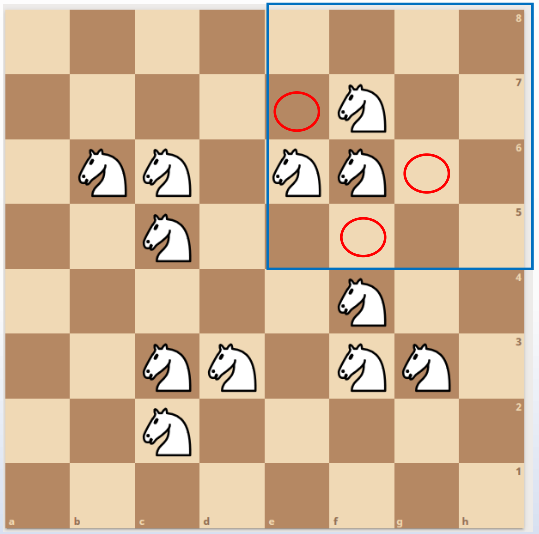
Kings - 9

- Of course, it is possible to form additional (but similar) solutions by shifting files and ranks – making sure the distance between kings is not more than two squares.

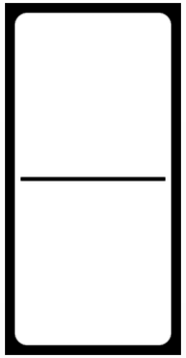


Knights - 12

- An interesting solution (I did not check whether it is unique except for symmetries, etc.) - each triple of knights covers a quarter of the board (for example, the “northeastern” quarter is marked by blue frame), with a little help from adjacent triples – for example, top-right triple “gets help” to cover squares g6, f5, e7 - marked in red.

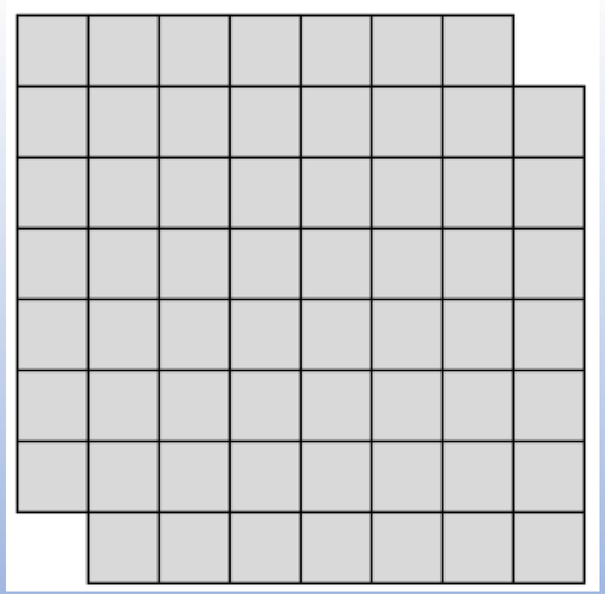


Problem #3
Tiling a board with dominoes

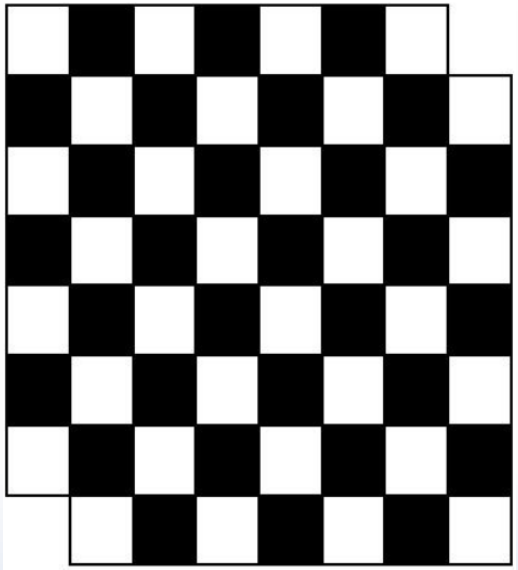


We take an 8x8 board and remove the two corners a1 and h8 from it, leaving 62 squares.

Can we fully tile this truncated board by 31 dominoes of size 2x1?



Tiling a board with dominoes
Solution



We take an 8x8 board and remove the two corners a1 and h8 from it, leaving 62 squares.
Can we fully tile this truncated board by 31 dominoes of size 2x1?

Answer:

The trick is to color the board as a chessboard. Then, we notice that in the truncated board there are 32 white squares, and only 30 black squares.

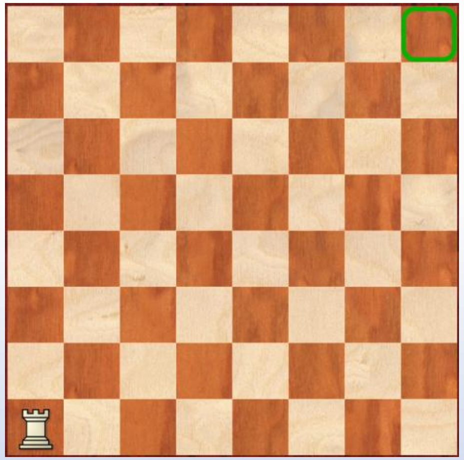
Because of the shape of the dominoes - when you place them, they cover one square of each color, and so you can't cover the truncated board that way, because the colors don't balance.

A follow-up question to the dominoes question

A rook starts moving from a1. The goal is to visit every square exactly once and finish on h8.

(Clarification - if we move Ra1-c1, then it is considered that we have also visited b1.)

Is it possible?



Answer:

No. Each time along rook's tour we jump from a black square to a white square and vice versa, alternatingly. Therefore, if our initial square is black - then the 64th and last square will be white, since we have passed 63 squares, and because it is an odd number, the color of the final square will be different from the initial one. Because h8 is black - such a tour is not possible.

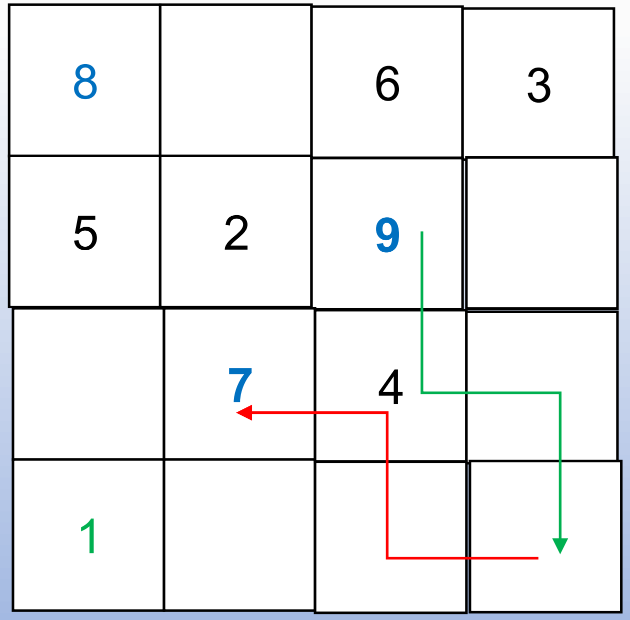
Problem #4
Knock-out tournament / loser gets eliminated

- Let's say 128 players (which is 2^7) gather for a chess tournament by the knock-out system / the loser get eliminated (for example: the World Cup!)
- How many games will be played in total until the winner is announced in the final?

- You can start and calculate that there will be 64 games in the first round, 32 in the second round, 16 in the third round, etc., and then add up the numbers (for those who know – a geometric sequence).
- Or, a much simpler way - notice that after each game a single player is eliminated, and since we start with 128 players and end with one winner - the number of games is 127.
  - And indeed 1+2+4+8+16+32+64 = 127

Problem #5
A knight tour

- We will examine different board sizes NxN, in which a knight starts at a1, and the goal is to visit every square exactly once.
  - 1x1 board - V   And even easy!
  - 2x2 ,3x3, 4x4 boards - X



A knight tour - board 5x5 and above

- For a 5x5 board and bigger – there is always a solution.
- It is interesting that to this day we do not know how many solutions there are on an 8x8 board, and we only know that there is a total of 26,534,728,821,064 different closed tours (it is enough that a small part of the solution be in reverse order to be counted as a different solution) – that is, ending on a1 (so the last step will be from b3 or c2).
- In mathematical research, an "upper bound" for the number of general solutions (in which you can finish on any square) was found - a 22-digit number...

- Such a tour is called Hamiltonian in the field of mathematics “graph theory”.

An amazing feature - the tour as a magic square

- On the right we have a knight tour, such that amazingly – the board is a magic square with the following properties:

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

- Every row adds up to 260
- Every column adds up to 260
- Each quarter of the board add up to 520 (twice 260…)
- Every "half column" – from row 1 to 4 or from 5 to 8 (e.g., a1-a2-a3-a4) – adds up to 130
- Every "half row" – from a-file to d-file or from e-file to h-file – adds up to 130
- Every 2x2 square originated in the corners adds up to 130
- However, it is not a real magic square - because the diagonals do not add up to 260. Such a square cannot be produced from a knight tour…

- Why do you think all the sums are 130 and its multiples? What property of the 8x8 board gives us to the number 260?

A knight tour - another version

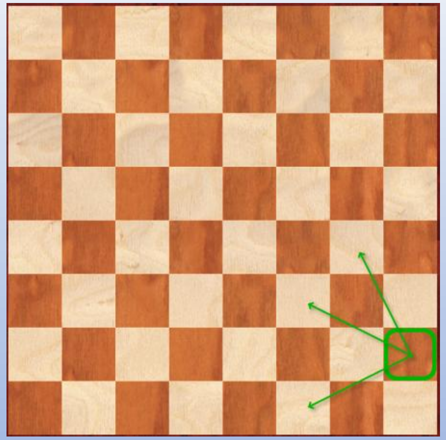
- Let's imagine that there is a bridge between every pair of squares which are a knight move apart.
- Is it possible to start at a1, cross all the bridges exactly once (if we crossed in a certain direction, we cannot cross the same bridge in the other direction) and return to a1?

(Note that here we are allowed to visit the same square more than once.)

Solution:

No! Note that a necessary condition for such a tour is that each square has an even number of bridges, since if we left a square through a certain bridge, we must return to it through another bridge, and because we must cross all the bridges (once) and return to a1, the number of bridges connected to each square must be even.

But, for example, there are 3 knight moves from h2 and so 3 bridges come out of it.

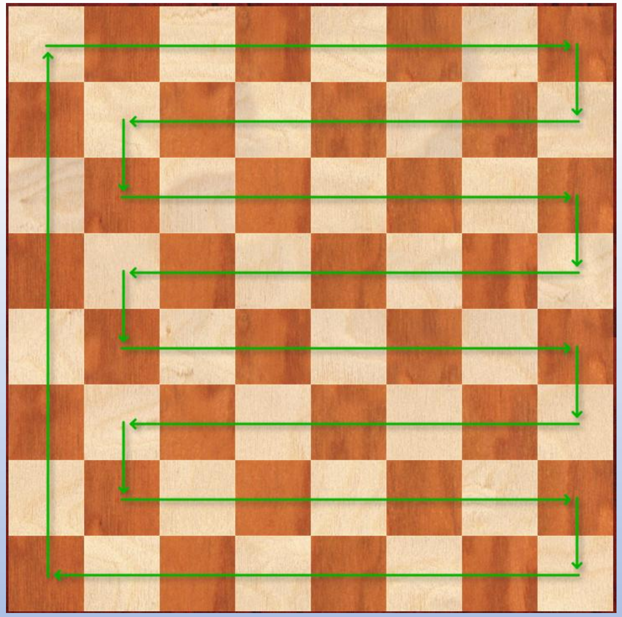


- Such a tour is called Eulerian in the field of mathematics “graph theory”,

 and it is known that it exists if the number of bridges from each square is even.

Tours of other pieces

- Bishop and pawn – no need to elaborate...
- Queen, Rook, King – similar simple solution…



Problem #6
Features of a board with numbered squares

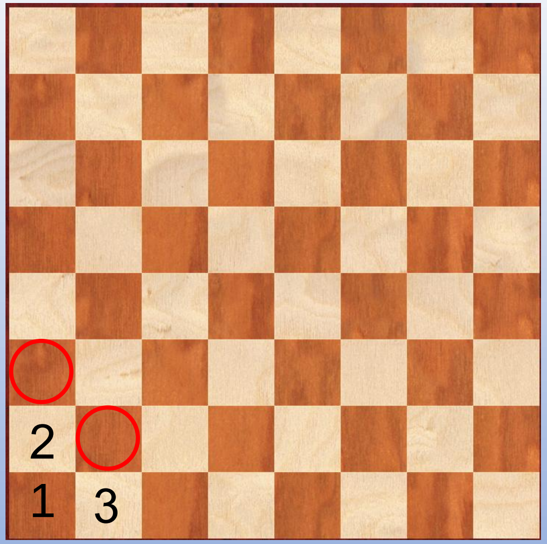
- We number the squares of the 8x8 board with numbers from 1 to 64 arbitrarily.
- Two squares are called neighboring if they share a common side. For example: b1 and b2 are neighbors.
- Question: Will there always be two neighboring squares with a difference of at least 3?

  At least 5?
- Additional question: What difference we can no longer guarantee (that is, in every numbering of the board)?

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

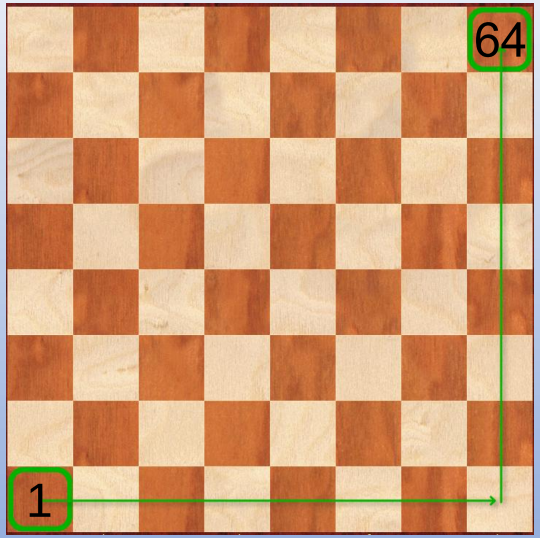
Features of a board with numbered squares

- We start by showing that we must have 2 neighboring squares with difference at least 3.
- Let's assume that this is not the case and notice that then 1 must be written in one of the corners, because every other square has at least three neighbors, and so one of the them will be at least 4.
- Then the two neighbors of 1 must be 2, 3.
- Now, the square with 2 has two empty neighbors, and so one of them will be at least 5.



Features of a board with numbered squares

- To show that there must be a difference of at least 5 is more subtle.
- Let's look on the shortest path between the squares where 1 and 64 are written, when by path we mean going via neighboring squares.
- It is easy to see that the length of this path is at most 14, with 1 and 64 being in opposite corners (say a1 and h8 as in the example).
- The sum of the differences (between neighboring squares) on this path is 63 (as we start at 1 and end at 64), and we make 14 steps when going from a1 to h8.
- As for each step there is a difference of squares that corresponds to it, from the generalized pigeonhole principle we know that there is a difference which is at least 63/14, that is, at least 5.



Features of a board with numbered squares

- We cannot always guarantee a difference of at least 9, as can be seen in the following simple example.
- It is known that we can always guarantee a difference of at least 8, and in an nxn board we can guarantee a difference of at least n.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Problem #7

Number of starting positions in Fischer Random chess

- As is known, in Fischer's chess one starts with an initial position in which the pieces in the back row are arranged randomly, under the following 2 constraints:
  - The two bishops must be on squares of different color.
  - The king must be placed between the 2 rooks.
- A natural question is: how many such legal positions are there? We will prove by a direct combinatorial counting that this number is 960.

Number of starting positions in Fischer Random chess – cont.

- For a positive integer n, the product of 1*2*…*n is denoted by n! – “n factorial”.
- Let’s start with a naive upper bound: if all 8 pieces were distinct, in how many ways could they be arranged in the back row without restrictions? We have 8 choices for a piece on a1, then 7 choice for a piece on b1, and so on. Thus, 8*7*…*1 = 8! possible arrangements.
- Now, as we have identical pieces, then, for example, an arrangement in which the first knight is on b1 and the second is on d1 is the same as one in which the second knight is on b1 and the first is on d1.
- The “trick” to calculating arrangements when there are identical objects is to treat them as different and then divide by the number of “internal arrangements”.
- If 2 squares were chosen for the knights, then there are 2!=2 ways to arrange them inside those squares, and these two arrangements are identical (if there were 3 identical pieces then the number of internal arrangements would be 3!=6). Thus, the number of possibilities for arranging pieces in the back row without restrictions is: 8!/(2!*2!*2!).

Number of starting positions in Fischer Random chess – cont.

- Now, let's calculate the number of positions in Fischer's chess.
- The restriction on the bishops is easy to deal with, because initially the positions of the bishops can be chosen in 4*4=16 ways.
- The more problematic restriction is regarding king + rooks. It seems that each case must be examined separately: placing the king on b1 and then examining in how many ways the rooks can be placed; then placing it on c1 and so on (and doing this taking into account the bishops' positions). This is a direct approach to counting; it is not difficult but contains many subcases.
- Key observation: choosing 3 squares for king + rooks **uniquely determines** their position inside those squares! In particular, we need to count in how many ways we can choose 3 squares out of 8. The problem is that we can choose squares of any color in this way, and then we will again have to examine cases for arranging the bishops.
- Therefore, first we choose the bishops' positions in 4*4=16 ways; 6 squares remain free.
- Now, there are 6 choices to position the queen; 5 squares remain free.
- We choose positions for the two knights: There are (5*4)/2 choices, when we divide by 2 because the knights are identical and so, for example, choosing a1 for the first knight and c1 for the second is identical to choosing c1 for the first and a1 for the second.
- Finally, only 3 squares left, and there the king and rooks will be placed – with the king being between the rooks!
- Therefore, the total number of possible arrangements is: 4*4*6*(5*4)/2 = 16*6*10 = 960.

Number of starting positions in Fischer Random chess – cont.

In the following Wikipedia articles you can read about a way to generate a random ordering using dice of different sizes ("dimensions"), as well as other interesting information.

- https://en.wikipedia.org/wiki/Chess960
- https://en.wikipedia.org/wiki/Fischer_random_chess_starting_position

A problem to think about at home

In a chess round-robin tournament of 8 players (each player plays against each other once), it is given that:

- At the end of the tournament, all players have a different score.
- The score of 2^{nd} place is equal to the sum of the points of places 5-8.

What was the result of the game between 3^{rd} place and 7^{th} place?

Extra material

Chess-Math playlist – great videos on Numberphile channel

- 8-queens problem
- Magic square of the knight tour
- and more

Thanks for listening!

Any questions: yevlev87@gmail.com